বিদ্যাসাগর বিশ্ববিদ্যালয়

VIDYASAGAR UNIVERSITY

Question Paper

B.Sc. Honours Examinations 2021

(Under CBCS Pattern)

Semester - III

Subject : MATHEMATICS

Paper : C 5 - T

Full Marks : 60

Time : 3 Hours

Candidates are required to give their answers in their own words as far as practicable.
The figures in the margin indicate full marks.

[THEORY OF REAL FUNCTIONS AND INTRODUCTION TO METRIC SPACE]

(Theory)

Group - A

1. Answer any ***four*** of the following questions : $12\times4=48$

(i) Let I be an interval. Define uniform continuity of a function on the interval I.

Prove that every uniform continuous function on I is continuous on I. Is the converse true?

Justify your answer. For a uniform continuous function f on I prove that if $\{x_n\}$ is a Cauchy sequence on I then $\{f(x_n)\}$ is a Cauchy sequence on $\mathbb{R}$.

2+3+3+4

(ii) (a) Let $f:[a,b] \rightarrow R$ be continuous on [a, b] and $f''(x)$ exists for all $x \in (a,b)$. Let $a < c < b$. Prove that there exists a point β in (a, b). Such that

$$f(c) = \frac{b-c}{b-a} f(a) + \frac{c-a}{c-b} f(b) + \frac{1}{2}(c-a)(c-b) f''(\beta).$$

(b) Let $f: R \rightarrow R$ be a differentiable function on R and $f'(x) > f(x)$ for all $x \in R$. If $f(0) = 0$, prove that $f(x) > 0$ for all $x > 0$.

(c) Use Taylor theorem to prove that $x - \frac{x^3}{6} < \sin x < x$ if $0 < x < \pi$.

4+4+4

(iii) (a) Verify Maclaurin's infinite series expansion of the following function on the indicated intervals.

$$\log(1+2x) = 2x - \frac{(2x)^2}{2} + \frac{(2x)^3}{3} - \ldots\ldots, \text{ for } -\frac{1}{2} < x < \frac{1}{2}$$

(b) Let ω denote the space of all sequences in K. Define $d(x,y) = \sum_{n=1}^{\infty} \frac{|x_n - y_n|}{2^n (1 + |x_n - y_n|)}$ where $x = \{x_n\}$ and $y = \{y_n\}$ are in ω. Prove that d defines a metric on ω.

(c) Find the points of discontinuity of the function $f(x) = (-1)^{[x]}, x \in R$. Examine the nature of discontinuity also.

5+5+2

(iv) (a) Show that in a discrete space X, every subset is open and closed.

(b) Let C denote the set of all complex numbers. Let us consider a mapping $d: C \times C \rightarrow R$ given by $d(z_1, z_2) = \begin{cases} |z_1 + z_2|, & \text{for all } z_1 \neq z_2 \\ 0 & \text{, otherwise} \end{cases}$.

Prove that (C, d) is a metric space.

5+7

(v) (a) Let the function f be continuous in the closed interval [a, b] and $f(a)f(b)<0$. Show that there exists at least one point $c, a<c<b$ such that $f(c)=0$.

(b) Let $f:[a,b]\to\mathbb{R}$ and $g:[a,b]\to\mathbb{R}$ be both continuous on [a, b]. Prove that the set $S=\{x\in[a,b]: f(x)\neq g(x)\}$ is an open set in but $S=\{x\in[a,b]: f(x)=g(x)\}$ is a closed set. 6+6

(vi) (a) Let a function f be defined on $\mathbb{R}$ by $f(x)=\begin{cases} x^2\sin\frac{1}{x^2}, & x\neq 0 \\ 0, & x=0\end{cases}$. Show that f is continuous and differentiable on $\mathbb{R}$ but the derive function $f'(x)$ is not continuous on $\mathbb{R}$.

(b) State Rolle's theorem for polynomial. Prove that if $\frac{a_0}{n+1}+\frac{a_1}{n}+.....+\frac{a_{n-1}}{2}+a_n=0$ then the equation $a_0x^n+a_1x^{n-1}+....+a_{n-1}x+a_n=0$ has at least one real root between 0 and 1.

(c) Let $D\subset\mathbb{R}$ and $f: D\to\mathbb{R}$ be a function. If c be an isolated point of D then prove that f is continuous at c. 5+(1+4)+2

(vii) (a) Let (X, d) be a metric space and $A\subset X$ then prove that the derived set $D(A)$ of the set A contains all its limit points. If $A, B\subset X$. Then prove that $D(A\cap B)\subset D(A)\cap D(B)$.

(b) Define open set. Prove that every open sphere is an open set.

(c) Define separable metric space with example. (3+2)+(1+4)+2

(viii) (a) Let l_p be the set of all those sequences $X=\{x_n\}$ of real or complex numbers such that $\sum_{n=1}^{\infty}|x_n|^p$ is convergent. Define a function $d: l_p\times l_p\to\mathbb{R}$ given by $d(x,y)=\left\{\sum_{n=1}^{\infty}|x_n-y_n|^p\right\}^{\frac{1}{p}}$ where $x=\{x_n\}, y=\{y_n\}\in l_p$. Prove that d is a metric on l_p.

(b) Let (*X*, *d*) be a metric space. Show that the function d_1 defined by $d_1(x,y)=\frac{d(x,y)}{1+d(x,y)}$ for all $x,y \in X$ is a metric on *X*.

(c) Define pseudo metric. 5+5+2

Group - B

2. Answer any ***six*** of the following questions : 2×6=12

(i) Prove that $\lim_{x\to 0} \cos\frac{1}{x}$ does not exists. 2

(ii) Let *f* is continuous on $[a,b]$ and $f(x)\in[a,b]$ for every $x\in[a,b]$ then prove that there exists a $c\in[a,b]$ such that $f(c)=c$. 2

(iii) Prove that a real valued function satisfying Lipschitz's condition on an interval *I* is uniformly continuous there.

(iv) Defined Lipschitz's function with example. 2

(v) Find the extreme values of the function $f(x)=x^{\frac{1}{x}}$ in its domain. 2

(vi) Prove that if *f* be defined for all real *x* such that $|f(x)-f(y)|<(x-y)^2$ then *f* is constant. 2

(vii) Define Pseudo metric with example. 2

(viii) Discuss the geometrical interpretation of MVT. 2

(ix) In a metric space (*X*, *d*) show that $d(x,y)\geq|d(x,z)-d(z,y)|$ for any three elements $x,y,z\in X$. 2

(x) Let *A* be a subset of a metric space (*X*, *d*). If *x* be a limit point of *A*, then prove that every neighbourhood of *x* contains infinite number of points of *A*. 2